Lyapunov Exponents and Modes for Hard Disk Systems

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Continuous vs. Discrete Maps

- Consider a general L-dimensional smooth dynamical system

$$\dot{\mathbf{\Gamma}} = \mathbf{F}(\mathbf{\Gamma}) \tag{1}$$

- Integrate to get

$$\mathbf{\Gamma}(t) = \Phi^t(\mathbf{\Gamma}(\mathbf{0})) \quad \Phi \text{ is the phase flow}$$

- Let $\mathbf{\Gamma}_s(t)$ be a perturbed trajectory of the original reference trajectory $\mathbf{\Gamma}(t)$ with parameter s s.t.

$$\lim_{s \to 0} \mathbf{\Gamma}_s(t) = \mathbf{\Gamma}(t)$$

Continuous vs. Discrete Maps (cont.)

- The tangent vector associated with this parametrization

$$\delta\mathbf{\Gamma}(t) = \lim_{s \to 0} \frac{\mathbf{\Gamma}_s(t) - \mathbf{\Gamma}(t)}{s}$$

- The equation of motion for this tangent vector is obtained by linearizing (1)

$$\delta\dot{\mathbf{\Gamma}} = \mathbf{D}(\mathbf{\Gamma}) \cdot \delta\mathbf{\Gamma}$$

  $\mathbf{D}(\mathbf{\Gamma}) = \partial\mathbf{F}/\partial\mathbf{\Gamma}$ is the system's Jacobian.

- A discrete map has the form

$$\mathbf{\Gamma}_f = \mathbf{M}(\mathbf{\Gamma}_i)$$

Hard Disk Dynamics

- The hard disk system is a *hybrid dynamical system*. The system exhibits continuous behavior interrupted at discrete times by a discrete collision map. All discussions concern the 2D version. (In 3D, hard spheres or balls are considered).
- The continuous portion of the dynamics evolves according to the simple relations

$$\begin{aligned} \dot{\mathbf{x}}^i &= \mathbf{v}^i \\ \dot{\mathbf{v}}^i &= \mathbf{0} \end{aligned}$$

where $\mathbf{x}, \mathbf{v}$ are 2D systems.

Hard Disk Dynamics

- When particles undergo collision, the colliding particles j, k velocities evolve according to the momentum-preserving rule

$$\mathbf{v}^j \mapsto \mathbf{v}^j + \frac{1}{\sigma^2}(\mathbf{v} \cdot \mathbf{x})\mathbf{x}$$
$$\mathbf{v}^k \mapsto \mathbf{v}^k - \frac{1}{\sigma^2}(\mathbf{v} \cdot \mathbf{x})\mathbf{x}$$

$\mathbf{x} = \mathbf{x}^k - \mathbf{x}^j, \mathbf{v} = \mathbf{v}^k - \mathbf{v}^j$ represent the difference vectors between the states of particles j, k. σ is the disk diameter.

Definition of Lyapunov Exponents and Modes

- The Lyapunov exponents of a trajectory with initial conditions $\mathbf{\Gamma}(0)$ and initial displacement $\delta\mathbf{\Gamma}(0)$ are defined as

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\delta\mathbf{\Gamma}(t)|}{|\delta\mathbf{\Gamma}(0)|}$$

- Due to Oseledec's multiplicative ergodic theorem, λ exists, and there are L orthonormal initial vectors $\delta\mathbf{\Gamma}_l(0)$ yielding a set of L exponents $\{\lambda_l\}$ (the Lyapunov spectrum of the system).
- The evolution of the $\delta\mathbf{\Gamma}_l(t)$ orthonormal vectors are known as the Lyapunov modes of the system. For large Lyapunov exponents, these vectors change rapidly. However, for lower exponents, the vectors converge; demonstrating global behavior of the system.

Description of Algorithm

- In the definition of the Lyapunov exponents, a large number of matrix (Jacobian) multiplications are required. These operations can cause numerical overflow (or underflow). Therefore, a more sophisticated technique is needed.
- At each iteration, we can renormalize the evolution of the vectors $\delta\mathbf{\Gamma}_l(t)$ using Gram-Schmidt. We just keep track of the renormalization factor.
- Using Gram-Schmidt is computationally expensive $(2n^3!)$. However, the exponents converge.

Application: Lorenz System

- The equations of motion for the Lorenz system are

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{aligned}$$

- $\sigma = 10, \rho = 28, \beta = \frac{8}{3}$

Lyapunov Exponents of the Lorenz System

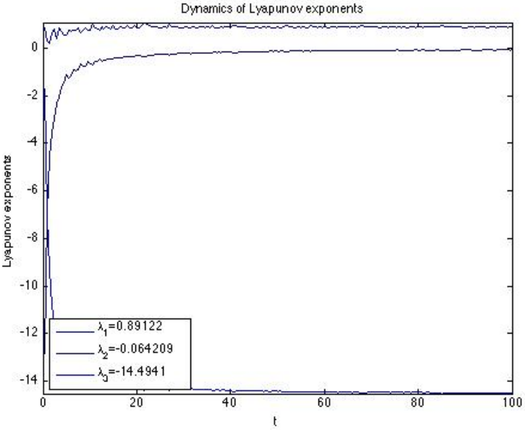

Figure: Lyapunov Exponents of the Lorenz System

Tangent Space Map for a Hard Disk System

- For a hard disk system, the perturbation vector after a collision is equal to

$$\delta\mathbf{\Gamma}_f = \frac{\partial\mathbf{M}}{\partial\mathbf{\Gamma}} \cdot \delta\mathbf{\Gamma}_i + \left[\frac{\partial\mathbf{M}}{\partial\mathbf{\Gamma}} \cdot \mathbf{F}(\mathbf{\Gamma}_i) - \mathbf{F}(\mathbf{M}(\mathbf{\Gamma}_i))\right]\delta\tau_c$$

- The Jacobian of the collision map M is differentiable in all coordinates (see defn from previous slides).
- $\delta\tau_c$ is the time delay associated with the difference in collision time between the reference and linearized trajectories.

Hard Disk Movie

Hard Disk Animation

- Growth of norm of separation of reference and satellite trajectories is determined by largest Lyapunov exponent.

Largest Lyapunov Exponent

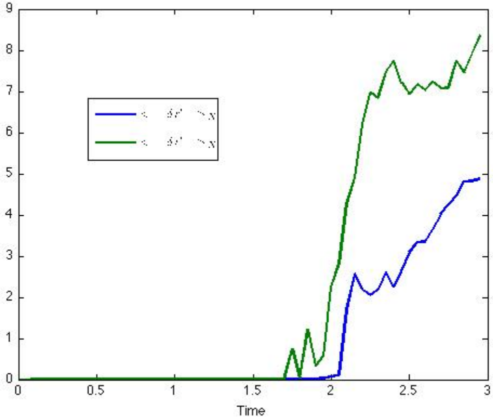

Figure: Largest Lyapunov Exponent of the Hard Disk System

Future Work

- Test algorithm and verify that the largest Lyapunov exponent for hard disk system converges to the same value as found from calculation from two trajectories.
- Fill in the complete Lyapunov spectrum.
- Verify against literature.

References

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